IDEAL CONVERGENCE OF BOUNDED SEQUENCES

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Abstract. We generalize the Bolzano-Weierstrass theorem (that every bounded sequence of reals admits a convergent subsequence) on ideal convergence. We show examples of ideals with and without the Bolzano-Weierstrass property, and give characterizations of BW property in terms of submeasures and extendability to a maximal P-ideal. We show applications to Rudin-Keisler and Rudin-Blass orderings of ideals and quotient Boolean algebras. In particular we show that an ideal does not have BW property if and only if its quotient Boolean algebra has a countably splitting family.

1. Introduction

An *ideal on* ω is a family of subsets of natural numbers ω closed under taking finite unions and subsets of its elements. In this paper we study ideal convergence ($\mathcal{I}$-convergence) of sequences defined on ω.

The notion of ideal convergence ($\mathcal{I}$-convergence) is a generalization of the notion of convergence (in the case of the ordinary convergence the ideal $\mathcal{I}$ is equal to the ideal of finite subsets of ω). It was first considered in the case of the ideal of sets of statistical density 0 by Steinhaus and Fast [9] (in such case ideal convergence is equivalent to the statistical convergence.) In its general form it appears in the work of Bernstein [4] (for maximal ideals) and Katětov [14], where both authors use dual notion of filter convergence. In the last few years it was rediscovered and generalized in many directions, see e.g. [2], [5], [6], [17], [21], [23].

By the well-known Bolzano-Weierstrass theorem any bounded sequence of reals admits a convergent subsequence. In other words, for any sequence $(x_n) \subset [0,1]$ there exists an infinite set A such that $(x_n) \restriction A$ is convergent. We consider a question if for given ideal $\mathcal{I}$ the assertion of the Bolzano-Weierstrass theorem is valid. We give a characterization of the Bolzano-Weierstrass property for some classes of ideals. We give these characterizations in the language of lower semicontinuous submeasures in Section 3. In Section 4 we show that this property of an ideal is connected with the possibility of its extending to a P-point of the Čech-Stone compactification $\beta\omega$.

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Some objects related to an ideal can be quite complex, for example its quotient Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}$ (see e.g. [7]). In Section 3 we show how the Bolzano-Weierstrass property distinguishes quotient Boolean algebras. In Section 5 we also investigate splitting families of quotient Boolean algebras.

Last we consider if the Bolzano-Weierstrass property is preserved by Rudin-Keisler-like orderings of ideals (Section 6).

2. Preliminaries

Our notation and terminology conforms to that used in the most recent set-theoretic literature. The cardinality of the set X is denoted by $|X|$. Following von Neumann, we identify a natural number N with the set $\{0, 1, \ldots, N-1\}$.

If not explicitly said we assume that an ideal is proper ($\neq \mathcal{P}(\omega)$) and contains all finite sets. By Fin we denote the ideal of all finite subsets of ω. We can talk about ideals on any other countable set by identifying this set with ω via a fixed bijection.

We say that $A \subset \omega$ is *$\mathcal{I}$-positive* if $A \notin \mathcal{I}$. An ideal $\mathcal{I}$ is *dense* if every infinite set has an infinite subset which is from the ideal.

For an ideal $\mathcal{I}$ on ω, $\mathcal{I}^*$ denotes the *dual filter*, i.e. the filter on ω consisting of the complements of elements of $\mathcal{I}$. For a set $A \notin \mathcal{I}$ we define an ideal $\mathcal{I} \restriction A = \{B \cap A : B \in \mathcal{I}\}$.

We say that $A \subset \omega$ is *$\mathcal{I}$-contained* in $B \subset \omega$ (B *$\mathcal{I}$-includes* A) for an ideal $\mathcal{I}$ and write $A \subset^{\mathcal{I}} B$ ($B \supset^{\mathcal{I}} A$) iff $A \setminus B \in \mathcal{I}$. We say that A is *almost contained* in B (B *almost contains* A) and write $A \subset^{\star} B$ ($B \supset^{\star} A$) if A is Fin-contained in B (B Fin-includes A, respectively).

A sequence $(x_n)_{n\in\omega}$ of reals is said to be *$\mathcal{I}$-convergent* to x ($x = \mathcal{I} - \lim x_n$) if and only if for each $\varepsilon > 0$

$$\{n \in \omega \ : \ |x_n - x| \geq \varepsilon\} \in \mathcal{I}.$$

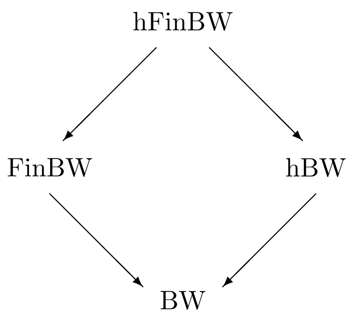
By an $\mathcal{I}$-subsequence of $(x_n)_{n\in\omega}$ we understand $(x_n) \restriction A$ for some $A \notin \mathcal{I}$. We will use the term subsequence instead of $\mathcal{I}$-subsequence if $\mathcal{I}$ is clear from the context. We say that a subsequence $(x_n) \restriction A$ is $\mathcal{I}$-convergent if it is $\mathcal{I} \restriction A$-convergent, i.e. $\{n \in A \ : \ |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for each $\varepsilon > 0$.

In the present paper we will consider the following properties of ideals on ω.

- **BW:** An ideal $\mathcal{I}$ satisfies BW (the *Bolzano-Weierstrass property*) if for any bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \notin \mathcal{I}$ such that $(x_n) \restriction A$ is $\mathcal{I}$-convergent.
- **FinBW:** An ideal $\mathcal{I}$ satisfies FinBW (the *finite Bolzano-Weierstrass property*) if for any bounded sequence $(x_n)_{n\in\omega}$ of reals there is $A \notin \mathcal{I}$ such that $(x_n) \restriction A$ is Fin-convergent.
- **hBW:** An ideal $\mathcal{I}$ satisfies hBW (the *hereditary Bolzano-Weierstrass property*) if for any bounded sequence $(x_n)_{n\in\omega}$ of reals and $A \notin \mathcal{I}$ there is $B \subset A, B \notin \mathcal{I}$ such that $(x_n) \restriction B$ is $\mathcal{I}$-convergent.
- **hFinBW:** An ideal $\mathcal{I}$ satisfies hFinBW (the *hereditary finite Bolzano-Weierstrass property*) if for any bounded sequence $(x_n)_{n\in\omega}$ of reals and $A \notin \mathcal{I}$ there is $B \subset A, B \notin \mathcal{I}$ such that $(x_n) \restriction B$ is Fin-convergent.

We will also write $\mathcal{I} \in$ BW ($\mathcal{I} \in$ hBW, $\mathcal{I} \in$ FinBW, $\mathcal{I} \in$ hFinBW) if $\mathcal{I}$ has the Bolzano-Weierstrass property ($\mathcal{I}$ has hBW property, $\mathcal{I}$ has FinBW property or $\mathcal{I}$ has hFinBW property, respectively.)

Clearly, the following diagram holds (where "$\mathcal{A} \to \mathcal{B}$" means "if $\mathcal{I} \in \mathcal{A}$ then $\mathcal{I} \in \mathcal{B}$".)



Note that if a sequence (x_n) has an $\mathcal{I}$-convergent subsequence $(x_n) \restriction A$ and $\mathcal{I} - \lim(x_n) \restriction A = x$ then x is an $\mathcal{I}$-cluster point of (x_n), i.e. for every $\varepsilon > 0$

$$\{n \ : \ |x_n - x| < \varepsilon\} \notin \mathcal{I},$$

but the opposite implication does not hold. Indeed, it is easy to see that every bounded sequence has an $\mathcal{I}$-cluster point but in Section 3 we show examples of ideals without the Bolzano-Weierstrass property. See also [10] for a discussion of $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points in the case of the ideal of sets of statistical density 0.

For two ideals $\mathcal{I}, \mathcal{J}$ define their *direct sum*, $\mathcal{I} \oplus \mathcal{J}$, to be the ideal on $\omega \times \{0,1\}$ given by

$$A \in \mathcal{I} \oplus \mathcal{J} \text{ iff } \{n \in \omega \ : \ \langle n, 0\rangle \in A\} \in \mathcal{I} \text{ and } \{n \in \omega \ : \ \langle n, 1\rangle \in A\} \in \mathcal{J}.$$

For $A \subset \omega \times \omega$ and $n \in \omega$ by A_n we denote the vertical section of A at n, i.e.

$$A_n = \{m \in \omega \ : \ \langle n, m\rangle \in A\}.$$

We define the *Fubini product*, $\mathcal{I} \times \mathcal{J}$ of $\mathcal{I}$ and $\mathcal{J}$ to be the ideal on $\omega \times \omega$ given by

$$A \in \mathcal{I} \times \mathcal{J} \text{ iff } \{n \in \omega \ : \ A_n \notin \mathcal{J}\} \in \mathcal{I}.$$

If $\mathcal{I}$ and $\mathcal{J}$ are two ideals on ω, we write $\mathcal{I} \leq_{RK} \mathcal{J}$ ($\mathcal{I}$ is below $\mathcal{J}$ with respect to the *Rudin-Keisler order*) if there exists a function $f \colon \omega \to \omega$ such that $A \in \mathcal{I}$ iff $f^{-1}(A) \in \mathcal{J}$. If a function f is finite-to-one then we write $\mathcal{I} \leq_{RB} \mathcal{J}$ ($\mathcal{I}$ is below $\mathcal{J}$ with respect to the *Rudin-Blass order*.)

We use the following notation for the ideals known from the literature. The ideal of sets of statistical density 0:

$$\mathcal{I}_d = \left\{ A \subset \omega : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\}.$$

The ideal of nowhere dense sets:

$$\mathrm{NWD}(\mathbb{Q}) = \{A \subset \mathbb{Q} \cap [0,1] : A \text{ is nowhere dense}\}.$$

The "empty ideal" $\{\emptyset\}$ (sometimes we write $\emptyset$ instead of $\{\emptyset\}$):

$$\text{“}\emptyset\text{”} = \{\emptyset\}.$$

Nevertheless this ideal does not contain all finite sets, its Fubini product with other ideals usually gives ideals which contain all finite sets.

2.1. **P-ideals.** An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence $(A_n)_{n\in\omega}$ of sets from $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subset^\star A$ for all n. In the sequel we will use the following property of P-ideals.

Theorem 2.1 ([17]). *If $\mathcal{I}$ is a P-ideal then a sequence $(x_n)_{n\in\omega}$ is $\mathcal{I}$-convergent iff subsequence $(x_n) \restriction F$ is Fin-convergent for some $F \in \mathcal{I}^*$.*

For every P-ideal $\mathcal{I}$ and $A \notin \mathcal{I}$, $\mathcal{I} \restriction A$ is also a P-ideal. It follows that every P-ideal with the Bolzano-Weierstrass property (with hBW property) has the finite Bolzano-Weierstrass property (hFinBW property, respectively).

2.2. **Analytic ideals.** Recall that by identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is F_σ (*analytic*) if it is an F_σ subset of the Cantor space (if it is a continuous image of a G_δ subset of the Cantor space, respectively.)

A map $\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ is a *submeasure on ω* if

$$\phi(\emptyset) = 0,$$
$$\phi(A) \le \phi(A \cup B) \le \phi(A) + \phi(B),$$

for all $A, B \subset \omega$. It is *lower semicontinuous* (in short lsc) if for all $A \subset \omega$ we have

$$\phi(A) = \lim_{n\to\infty} \phi(A \cap n).$$

For any lower semicontinuous submeasure on ω, let $\|\cdot\|_\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ be the submeasure defined by

$$\|A\|_\phi = \limsup_{n\to\infty} \phi(A \setminus n) = \lim_{n\to\infty} \phi(A \setminus n),$$

where the second equality follows by the monotonicity of ϕ. Let

$$\mathrm{Exh}(\phi) = \left\{A \subset \omega \ : \ \|A\|_\phi = 0\right\},$$
$$\mathrm{Fin}(\phi) = \{A \subset \omega \ : \ \phi(A) < \infty\}.$$

It is clear that $\mathrm{Exh}(\phi)$ and $\mathrm{Fin}(\phi)$ are ideals (not necessarily proper) for an arbitrary submeasure ϕ.

All analytic P-ideals are characterized by the following theorem of Solecki.

Theorem 2.2 ([25]). *The following conditions are equivalent for an ideal $\mathcal{I}$ on ω.*

1. *$\mathcal{I}$ is an analytic P-ideal;*
2. *$\mathcal{I} = Exh(\phi)$ for some lower semicontinuous submeasure ϕ on ω.*

Moreover, for F_σ ideals the following characterization holds.

Theorem 2.3 ([19]). *The following conditions are equivalent for an ideal $\mathcal{I}$ on ω.*

1. *$\mathcal{I}$ is an F_σ ideal;*
2. *$\mathcal{I} = Fin(\phi)$ for some lower semicontinuous submeasure ϕ on ω.*

In [7], Farah introduced a new class of analytic P-ideals. Assume that I_n are pairwise disjoint intervals on ω, and μ_n is a measure that concentrates on I_n. Then $\phi = \sup_n \mu_n$ is a lower semicontinuous submeasure and $\mathcal{Z}_\mu = \mathrm{Exh}(\phi)$ is called a *density ideal.*

Some interesting subclass of density ideals, so called Erdős-Ulam ideals, was introduced by Just and Krawczyk. Instead of original definition of these ideals we include here a very useful characterization of Erdős-Ulam ideals given by Farah:

Theorem 2.4 ([7])**.** *A dense density ideal $\mathcal{Z}_\mu$ is an Erdős-Ulam ideal if and only if for every choice of μ_n and I_n $(n \in \omega)$ we have* $\limsup_n \mu_n(I_n) < \infty$.

Later, Farah [8] introduced generalized density ideals. Assume that I_n are pairwise disjoint finite intervals on ω, and ϕ_n is a submeasure on I_n. Assume moreover that $\limsup_n at^+(\phi_n) = 0$ (where $at^+(\phi) = \sup_i \phi(\{i\})$). Then the ideal

$$\mathcal{Z}_\phi = \left\{ A \ : \ \limsup_{n\to\infty} \phi_n (A \cap I_n) = 0 \right\}$$

is a *generalized density ideal* defined by a sequence of submeasures.

2.3. **BW-like properties in topological spaces.** In the sequel we assume all our spaces to be Hausdorff. The ideal convergence can be defined for topological spaces in the natural way. Then we also define BW, FinBW, hBW and hFinBW properties for a pair $(X, \mathcal{I})$, where X is a topological space and $\mathcal{I}$ is an ideal. We say that $(X, \mathcal{I})$ satisfies BW if every sequence in X has an $\mathcal{I}$-convergent $\mathcal{I}$-subsequence (FinBW, hBW and hFinBW are defined analogously).

One can easily see that if Y is a continuous image of X (or Y is a closed subset of X) then

$$(X, \mathcal{I}) \in \mathrm{BW} \implies (Y, \mathcal{I}) \in \mathrm{BW}.$$

Thus, the following are equivalent:

- $\mathcal{I} \in \mathrm{BW}$,
- $([0,1], \mathcal{I}) \in \mathrm{BW}$,
- $(2^\omega, \mathcal{I}) \in \mathrm{BW}$,
- $(X, \mathcal{I}) \in \mathrm{BW}$ for some uncountable compact metric space X.

Clearly, the above properties hold for FinBW, hBW and hFinBW as well.

The assumption that a compact space X is metric (in the equivalent form of BW) cannot be dropped. In [16] Kojman constructed a compact space X such that $(X, \mathcal{W})$ does not satisfy FinBW whereas the ideal $\mathcal{W}$ is F_σ so it satisfies FinBW (see Proposition 3.4).

However, it is not difficult to check that if $\mathcal{I}$ is a maximal ideal then $(X, \mathcal{I})$ satisfies BW for every compact space X. Recall also that in papers [16], [15] Kojman gives sufficient conditions on X to have $(X, \mathcal{I}) \in \mathrm{FinBW}$ for two ideals defined by some combinatorial conditions. So, there are a number of natural questions about a characterization of BW-like properties in topological spaces. For example, by Theorem 2.1 BW and FinBW properties are equivalent for P-ideals, but it is not the case for an arbitrary topological space (since in $\beta\omega$ there are no non-trivial Fin-convergent sequences, $(\beta\omega, \mathcal{I}) \in \mathrm{BW} \setminus \mathrm{FinBW}$ for any maximal ideal $\mathcal{I}$—even if $\mathcal{I}$ is a P-ideal; note that it is not a ZFC-example—see comments before Theorem 4.2.)

Such questions are not in the scope of this paper, so we leave it for further study.

3. Basic properties

By the Bolzano-Weierstrass theorem the ideal of finite subsets of ω satisfies hFinBW. Every maximal ideal satisfies hBW but usually (if it is not a P-ideal) does not satisfy FinBW. In [10, Example 3] Fridy has shown that the ideal $\mathcal{I}_d$ does not satisfy BW. (Below we will show that every Erdős-Ulam ideal does not satisfy the Bolzano-Weierstrass property.)

Let $x_q = q$ be a sequence on $\mathbb{Q} \cap [0,1]$ and $A \notin \mathrm{NWD}(\mathbb{Q})$. Since there exists a nonempty open set U in which A is dense, $(x_q) \restriction A$ cannot be $\mathrm{NWD}(\mathbb{Q})$-convergent. So, we have

Example 3.1. The ideal $\mathrm{NWD}(\mathbb{Q})$ does not satisfy BW.

Now we show how to build examples of ideals with or without BW. First of all, it is easy to see that $\mathcal{I} \oplus \mathcal{J}$ satisfies BW iff $\mathcal{I}$ or $\mathcal{J}$ satisfies BW. Moreover

Proposition 3.2. *Assume $\mathrm{Fin} \subset \mathcal{I}$. $\mathcal{I} \times \mathcal{J}$ satisfies BW (hBW) iff $\mathcal{I}$ satisfies BW (hBW).*

Proof. "$\Rightarrow$". Let (x_n) be a bounded sequence on ω, and let $x_{n,m} = x_n$ be a sequence on $\omega \times \omega$. For every $A \notin \mathcal{I} \times \mathcal{J}$ if $(x_{n,m}) \restriction A$ is $\mathcal{I} \times \mathcal{J}$-convergent to x, then $(x_n) \restriction B$ is $\mathcal{I}$-convergent to x, where

$$B = \{n \in \omega : A_n \notin \mathcal{J}\}.$$

"$\Leftarrow$". To prove converse suppose that $(x_{n,m}) \subset [0,1]$ is a sequence on $\omega \times \omega$. For each $n \in \omega$ and $k \in \{0, \ldots, 2^n - 1\}$ define

$$A_{n,k} = \left\{ i \in \omega \ : \ x_{n,i} \in \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right] \right\}$$

For each $n \in \omega$ let k_n be such that $A_{n,k_n} \notin \mathcal{J}$. Moreover, let $y_n = (2k_n + 1)/2^{n+1}$. If $B \notin \mathcal{I}$ is such that $(y_n) \restriction B$ is $\mathcal{I}$-convergent to some y, then $(x_{n,m}) \restriction C$ is $\mathcal{I} \times \mathcal{J}$-convergent to y, where

$$C = \bigcup_{i \in B} \{i\} \times A_{i,k_i}.$$

The case of hBW is proved analogously. □

The assumption on $\mathcal{I}$ in the above proposition is necessary since $\emptyset \times \mathcal{J} \in \mathrm{BW} \iff \mathcal{J} \in \mathrm{BW}$.

In the definition of Fubini product $\mathcal{I} \times \mathcal{J}$ one can replace $\mathcal{J}$ by a family of ideals $\{\mathcal{I}_n\}_{n \in \omega}$ to get the *countable Fubini product* i.e. the ideal of all sets $A \subset \omega \times \omega$ for which

$$\{n \in \omega \ : \ A_n \notin \mathcal{I}_n\} \in \mathcal{I}.$$

Using almost the same arguments as in the proof of Proposition 3.2 it can be shown that this ideal satisfies BW (hBW) iff $\mathcal{I}$ satisfies BW (hBW). However, assertion of Proposition 3.2 does not hold for FinBW and hFinBW property. Indeed, consider $\mathcal{I} = \mathcal{J} = \mathrm{Fin}$, $x_{n,m} = 1/n$ $(n, m \in \omega)$ and fix an $A \notin \mathcal{I} \times \mathcal{J}$. We may assume that A_n is infinite for every n. For any $g \in [0,1]$ and $\varepsilon > 0$ the set $\{(n,m) \in A \ : \ |x_{n,m} - g| \geq \varepsilon\}$ is either empty or contains A_n (so it is infinite) for some $n \in \omega$. Thus $\mathrm{Fin} \times \mathrm{Fin}$ does not satisfy FinBW property.

Farah in paper [8] introduced the notion of $\mathcal{I}$-small sets: we say that a set $A \subset \omega$ is *$\mathcal{I}$-small* if there are sets A_s $(s \in 2^{<\omega})$ such that for all s we have

(S1) $A_\emptyset = A$,
(S2) $A_s = A_{s^\frown 0} \cup A_{s^\frown 1}$,
(S3) $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$, and
(S4) for every $b \in 2^\omega$ and $X \subset \omega$, if $X \setminus A_{b \restriction n} \in \mathcal{I}$ for all n, then $X \in \mathcal{I}$.

It is known that all $\mathcal{I}$-small sets form an ideal $\mathcal{S}_\mathcal{I}$ that includes $\mathcal{I}$.

Proposition 3.3. *A set $A \notin \mathcal{S}_\mathcal{I}$ if and only if $\mathcal{I} \restriction A \in BW$.*

Proof. "$\Rightarrow$". Suppose that $A \notin \mathcal{S}_\mathcal{I}$ and $(x_n)_{n\in A} \subset 2^\omega$. For each $s \in 2^{<\omega}$ let $A_s = \{n \in A \ : \ s \subset x_n\}$. Since A is not $\mathcal{I}$-small, there exist $X \notin \mathcal{I}$ and $b \in 2^\omega$ such that $X \setminus A_{b\restriction n} \in \mathcal{I}$ for each n. Clearly $(x_n) \restriction X$ is $\mathcal{I}$-convergent to b.

"$\Leftarrow$". Suppose that $A \in \mathcal{S}_\mathcal{I}$. Let $\{A_s\}_{s\in 2^{<\omega}}$ be a family of sets fulfilling conditions (S1)–(S4) from the definition of $\mathcal{I}$-small set. Observe that for each $n \in A$ there is exactly one $x_n \in 2^\omega$ such that $n \in A_{x_n\restriction l}$ for every $l \in \omega$. For the sake of contradiction suppose that there is an $x \in 2^\omega$ and $X \subset A$, $X \notin \mathcal{I}$ such that $(x_n)_n \restriction X$ is $\mathcal{I}$-convergent to x. Since for each $l \in \omega$

$$X \setminus A_{x\restriction l} \subset \left\{n \in X \ : \ |x - x_n| \geq \frac{1}{2^l}\right\} \in \mathcal{I},$$

by the condition (S4) $X \in \mathcal{I}$—a contradiction. □

By [8, Lemma 3.2] any isomorphism between quotient Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ sends the equivalence classes of $\mathcal{I}$-small sets into the equivalence classes of $\mathcal{J}$-small sets, so if Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ are isomorphic then $\mathcal{I} \in$ BW if and only if $\mathcal{J} \in$ BW.

By [8, Proposition 3.3 (1)] we know that $\mathcal{S}_{\mathcal{Z}_\mu} = \mathcal{P}(\omega)$ for every Erdős-Ulam ideal $\mathcal{Z}_\mu$. Hence by Proposition 3.3 no Erdős-Ulam ideal satisfies BW. Moreover, [8, Proposition 3.3 (2)] says that if $\limsup_n \mu_n(I_n) = \infty$ then $\omega \notin \mathcal{S}_{\mathcal{Z}_\mu}$, so those ideals satisfy BW. Finally, by Theorem 2.4 we see that a density ideal does not satisfy BW if and only if it is an Erdős-Ulam ideal.

By [8, Proposition 3.3 (3)] if $\mathcal{Z}_\mu$ is a density ideal then there is $\mathcal{Z}_\mu$-positive set A which belongs to $\mathcal{S}_{\mathcal{Z}_\mu}$ hence $\mathcal{Z}_\mu \restriction A \notin$ BW. So by Proposition 3.3 no density ideal satisfies hBW.

A large subclass of generalized density ideals, *LV-ideals*, was introduced by Louveau and Veličković [18]. By [8, Proposition 3.3 (4)] we have $\mathcal{S}_{LV} = LV$. Hence Proposition 3.3 shows that $LV \in$ hBW.

One may ask if the BW and the hBW are the same property (up to the restriction), i.e. if every BW ideal is a direct sum of two ideals with at least one of them possessing hBW property (the reverse implication is obviously true.) By [8, Proposition 3.3 (3)] if $\mathcal{Z}_\mu$ is a density ideal then every $\mathcal{Z}_\mu$-positive set A contains a positive subset that belongs to $\mathcal{S}_{\mathcal{Z}_\mu}$. Thus no restriction of a density ideal with the Bolzano-Weierstrass property has the hereditary Bolzano-Weierstrass property.

Proposition 3.4. *Every F_σ ideal satisfies hFinBW property.*

Proof. Let $\mathcal{I}$ be an F_σ ideal, $\mathcal{I} = \mathrm{Fin}(\phi)$ for a lower semicontinuous submeasure ϕ. Let $(x_n)_n \subset 2^\omega$ and $A \notin \mathcal{I}$. For each $s \in 2^{<\omega}$ define

$$A_s = \{n \in A \ : \ s \subset x_n\} \text{ and } T = \left\{s \in 2^{<\omega} \ : \ A_s \notin \mathcal{I}\right\}.$$

Since $T \subset 2^{<\omega}$ is a tree of height ω such that all levels of T are finite, by König's lemma there is a $b \in 2^\omega$ such that $b \restriction n \in T$ for each n.

Note that $A_{b\restriction n}$ $(n \in \omega)$ is a decreasing sequence of $\mathcal{I}$-positive sets. Since ϕ is lsc there are finite sets $B_n \subset A_{b\restriction n}$ such that $\phi(B_n) > n$ for each n. Put $B = \bigcup_{n\in\omega} B_n$. Clearly $\phi(B) = \infty$, and so $B \notin \mathcal{I}$. Moreover, for every $n \in \omega$ $B \subset^\star A_{b\restriction n}$, so $(x_n) \restriction B$ is Fin-convergent to b. □

The argument above shows a bit more. In [13, Corollary 2.4] it was proved that the following are equivalent:

(1) $\mathcal{P}(\omega)/\mathcal{I}$ is countably saturated;

(2) for every sequence $B_0 \supset B_1 \supset \cdots$ of $\mathcal{I}$-positive sets there is an $\mathcal{I}$-positive set B with $B \subset^{\mathcal{I}} B_n$ for each n.

Using (2) in the last paragraph of the proof of Proposition 3.4 we get

Proposition 3.5. *If $\mathcal{P}(\omega)/\mathcal{I}$ is countably saturated then $\mathcal{I} \in hBW$.*

By a result of Just and Krawczyk ([12]) all F_σ ideals have countably saturated quotients. Farah in [8, Theorem 6.3] gives more examples of ideals with countably saturated quotients (e.g. Fin × Fin). The above proposition cannot be reversed, since $\emptyset \times \mathrm{Fin} \in \mathrm{hBW}$ whereas its quotient is not countably saturated.

By Theorem 2.2 the following conditions characterize the BW property for all analytic P-ideals.

Theorem 3.6. *Let ϕ be a lower semicontinuous submeasure. The following conditions are equivalent.*

(1) *The ideal $Exh(\phi)$ satisfies BW.*
(2) *There is $\delta > 0$ such that for any partition $A_1, A_2, \ldots, A_N$ of ω there exists $i \leq N$ with $\|A_i\|_\phi \geq \delta$.*

Proof. Let $\mathcal{I} = \mathrm{Exh}(\phi)$.

"$\neg(2) \Rightarrow \neg(1)$" Suppose that for each $\delta > 0$ there are $A_1, \ldots, A_{N(\delta)}$ such that $\omega = A_1 \cup \cdots \cup A_{N(\delta)}$ and $\|A_i\|_\phi < \delta$ for every $i \leq N(\delta)$. We can find a family A_s ($s \in 2^{<\omega}$) such that $A_\emptyset = \omega$, A_s fulfills conditions (S1)–(S3) from the definition of $\mathcal{I}$-small set and for every $s \in 2^\omega$ $\|A_{s\restriction n}\|_\phi \to 0$ if $n \to \infty$.

Let $b \in 2^\omega$ and $X \subset \omega$ be such that $\|X \setminus A_{b\restriction n}\|_\phi = 0$ for all n. Since $\|\cdot\|_\phi$ is subadditive, $\|X\|_\phi = 0$. Thus ω is $\mathcal{I}$-small, and so $\mathcal{I} \notin \mathrm{BW}$.

"$(2) \Rightarrow (1)$". For the sake of contradiction suppose that $\omega \in \mathcal{S}_\mathcal{I}$ and for every $A_1, \ldots, A_N$ such that $A_1 \cup \cdots \cup A_N = \omega$ there is an $i \leq N$ with $\|A_i\|_\phi \geq \delta$. Let A_s ($s \in 2^{<\omega}$) be a family of sets as in the definition of $\mathcal{I}$-small set (with $A_\emptyset = \omega$). Let

$$T = \left\{ s \in 2^{<\omega} \ : \ \|A_s\|_\phi \geq \delta \right\}.$$

We see that $T \subset 2^{<\omega}$ is a tree of height ω such that all levels of T are finite. Let $b \in 2^\omega$ be such that $b \restriction n \in T$ (it exists by König's lemma). Since ϕ is lsc, for each n there is a finite set $X_n \subset A_{b\restriction n} \setminus n$ such that $\phi(X_n) > \delta - \frac{1}{n}$. Let $X = \bigcup_{n\in\omega} X_n$. Clearly $X \setminus A_{b\restriction n} \in \mathrm{Fin} \subset \mathcal{I}$, but $\|X\|_\phi \geq \delta$, and so $\omega \notin \mathcal{S}_\mathcal{I}$—a contradiction. □

4. Extensions

Proposition 4.1. *If an ideal $\mathcal{I}$ can be extended to an ideal $\mathcal{J}$ satisfying FinBW then $\mathcal{I}$ satisfies FinBW. Thus, if an ideal $\mathcal{I}$ can be extended to a P-ideal satisfying BW then $\mathcal{I}$ satisfies FinBW property.*

Proof. Let (x_n) be a bounded sequence. There is $A \notin \mathcal{J}$ such that $(x_n)_n \restriction A$ is Fin-convergent. Since $\mathcal{I} \subset \mathcal{J}$, $A \notin \mathcal{I}$. The second part of the proposition follows from the first one and the fact that every P-ideal with the BW property has the finite Bolzano-Weierstrass property. □

Theorem 3.6 gives us a characterization of BW property in the class of all analytic P-ideals. Below we give two alternative characterizations of BW property within this class of ideals. The second one needs some extra set-theoretic assumptions—recall that Shelah ([24]) showed that it is consistent that there are no maximal P-ideals, so in Theorem 4.3 the additional set-theoretic assumption like CH is needed.

Theorem 4.2. *Let $\mathcal{I}$ be an analytic P-ideal. Then $\mathcal{I}$ has the Bolzano-Weierstrass property if and only if $\mathcal{I}$ can be extended to a proper F_σ ideal.*

Theorem 4.3. *Assume Continuum Hypothesis. Let $\mathcal{I}$ be an analytic P-ideal. Then $\mathcal{I}$ has the Bolzano-Weierstrass property if and only if $\mathcal{I}$ can be extended to a maximal P-ideal.*

The rest of this section will be devoted to the proof of Theorem 4.2. Theorem 4.3 follows from Proposition 4.1 ("$\Leftarrow$"), and Theorem 4.2 with Lemma 4.4 ("$\Rightarrow$").

Proof of Theorem 4.2. The part "$\Leftarrow$" follows from Propositions 3.4 and 4.1.

To show the part "$\Rightarrow$" suppose that $\mathcal{I} = \mathrm{Exh}\,(\phi)$, where ϕ is a lower semicontinuous submeasure, satisfies BW. By Theorem 3.6 there is $\delta > 0$ such that for every $N \in \omega$ and $k \in \omega$ and every $A_1, A_2, \ldots, A_N$ with $A_1 \cup A_2 \cup \cdots \cup A_N = \omega$ there is an $i \leq N$ with $\phi(A_i \setminus k) > \delta$.

We will say that $\{F_1, \ldots, F_N\}$ is an *(N, δ)-partition of the set $A \subset \omega$* if $F_1 \cup \cdots \cup F_N = A$ and $\phi(F_i) \leq \delta$ for every $i \leq N$. Let

$$\mathcal{I}_\delta = \{A \subset \omega \ : \ (\exists N, k \in \omega)\,(\forall n \in \omega)\,(\exists \mathcal{F})\ \mathcal{F} \text{ is } (N,\delta)\text{-partition of } A \cap [k,n]\}\,.$$

We claim that $\mathcal{I}_\delta$ is a proper F_σ extension of $\mathcal{I}$. The inclusion $\mathcal{I} \subset \mathcal{I}_\delta$ follows from the fact that if $A \in \mathcal{I}$ then $\|A\|_\phi = 0$. We show below that $\omega \notin \mathcal{I}_\delta$.

For the sake of contradiction suppose that there is $N \in \omega$ and $k \in \omega$ such that for every $n \in \omega$ there is an (N, δ)-partition of $[k, n]$. For each n let

$$T_n = \left\{f\colon\ [k,n] \to N \ : \ \left\{f^{-1}\left(\{i\}\right)\right\}_{i \leq N} \text{ is } (N,\delta)\text{-partition of } [k,n]\right\}.$$

Then $(\bigcup_{n\in\omega} T_n, \subset)$ is a tree such that its height is ω and every level is finite. Applying König's lemma we get a path $B = \{f_n \in T_n \ : \ n \in \omega\}$ through T. Let $g = \bigcup_{n\in\omega} f_n$. Then $g : [k, \infty) \to N$ and there is $i \leq N$ with $\phi(g^{-1}(\{i\})) > \delta$. By lsc of ϕ, there is $n \in \omega$ with $\phi(g^{-1}(\{i\}) \cap [k, n]) > \delta$. Then $f_n = g \restriction [k, n]$ hence $\phi(f_n^{-1}(\{i\})) > \delta$. But $f_n \in T_n$, a contradiction.

Finally, we have to show that $\mathcal{I}_\delta$ is F_σ. It follows from a standard quantifier-counting argument and the fact that there are only finitely many partitions of $[k, n]$ (let alone (N, δ)-partitions). $\square$

Remark. The ideal $\mathcal{S}_\mathcal{I}$ (of all $\mathcal{I}$-small sets) extends $\mathcal{I}$, moreover $\mathcal{S}_\mathcal{I}$ is proper iff $\mathcal{I}$ satisfies BW. However it is not F_σ in general. For example, for dense density ideals it is F_σ, but for $\mathcal{I} = \emptyset \times \mathrm{Fin}$ we have $\mathcal{S}_{\emptyset\times\mathrm{Fin}} = \emptyset \times \mathrm{Fin}$ which is not F_σ ([8]).

Using the standard argument (see e.g. [22]) we can prove

Lemma 4.4. *Assume Continuum Hypothesis. Every F_σ ideal can be extended to a maximal P-ideal.*

5. Splitting numbers

Let $\mathbb{B}$ be Boolean algebras. A subset $S \subset \mathbb{B}$ *splits* $\mathbb{B}$ if for every nonzero $b \in \mathbb{B}$ there is $s \in S$ such that $b \cdot s \neq 0$ and $b - s \neq 0$. By $\mathfrak{s}(\mathbb{B})$ we denote the smallest cardinality of a splitting family:

$$\mathfrak{s}(\mathbb{B}) = \min\{|S| \ : \ S \text{ splits } \mathbb{B}\}$$

In case of $\mathbb{B} = \mathcal{P}(\omega)/\mathrm{Fin}$ we get the well known splitting number $\mathfrak{s}$. Note that the cardinal $\mathfrak{s}(\mathbb{B})$ is well-defined iff $\mathbb{B}$ is atomless. And then we have $\mathfrak{s}(\mathbb{B}) \geq \omega$. (See e.g. [20] for more information on the splitting number and other cardinal invariants defined for Boolean algebras.)

Theorem 5.1. *A Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}$ has a countable splitting family if and only if the ideal $\mathcal{I}$ does not satisfy BW.*

Proof. "⇒". Let $\{[S_n] : n \in \omega\}$ be a splitting family for $\mathcal{P}(\omega)/\mathcal{I}$. We will construct a family $\{A_s : s \in 2^{<\omega}\}$ which satisfies conditions (S1)–(S4) (with $A_\emptyset = \omega$) from the definition of $\mathcal{I}$-small sets. By Proposition 3.3 we get $\mathcal{I} \notin \mathrm{BW}$.

Let $A_\emptyset = \omega$. Let $n_\emptyset$ be the smallest n such that S_n splits ω. Let $A_0 = \omega \cap S_{n_\emptyset}$ and $A_1 = \omega \setminus S_{n_\emptyset}$. Suppose that we have already constructed A_s for $s \in 2^n$. Take $s \in 2^n$. Let n_s be the smallest n such that S_n splits A_s. Let $A_{s\frown 0} = A_s \cap S_{n_s}$ and $A_{s\frown 1} = A_s \setminus S_{n_s}$.

Now we have to show that $\{A_s : s \in 2^{<\omega}\}$ satisfies (S1)–(S4). (S1)–(S3) are obvious. To show (S4) let $b \in 2^\omega$ and $X \subset \omega$ be such that $X \setminus A_{b\restriction n} \in \mathcal{I}$ for every $n \in \omega$. Suppose that $X \notin \mathcal{I}$. Let n_X be the smallest $n \in \omega$ such that S_n splits X. Notice that S_{n_X} splits $A_{b\restriction n}$ for every $n \in \omega$. Hence there is $k \leq n_X$ such that $S_{n_{b\restriction k}} = S_{n_X}$. Then either $A_{b\restriction k+1} = A_{b\restriction k} \cap S_{n_X}$ or $A_{b\restriction k+1} = A_{b\restriction k} \setminus S_{n_X}$. In both cases S_{n_X} does not split $A_{b\restriction k+1}$, a contradiction.

"⇐". By Proposition 3.3 there is a family $\{A_s : s \in 2^{<\omega}\}$ which shows that ω is $\mathcal{I}$-small. We will show that $\{[A_s] : s \in 2^{<\omega}\}$ is a splitting family. Suppose the contrary. Then there is a set $X \notin \mathcal{I}$ such that $X \cap A_s \in \mathcal{I}$ or $X \setminus A_s \in \mathcal{I}$ for every $s \in 2^{<\omega}$. On the other hand $\omega = \bigcup\{A_s : s \in 2^n\}$ for every $n \in \omega$, hence for every $n \in \omega$ there is $s \in 2^n$ with $X \setminus A_s \in \mathcal{I}$. Now, by König's lemma there is $x \in 2^\omega$ with $X \setminus A_{x\restriction n} \in \mathcal{I}$ for every $n \in \omega$. From the definition of the family $\{A_s : s \in 2^{<\omega}\}$ we have that $X \in \mathcal{I}$, a contradiction. □

Remark. The splitting numbers $\mathfrak{s}(\mathcal{P}(\omega)/\mathcal{I}_d)$ and $\mathfrak{s}(\mathcal{P}(\mathbb{Q})/\mathrm{NWD}(\mathbb{Q}))$ were already calculated in [11] and [1], respectively.

6. Orderings

Theorem 6.1. *Suppose that $\mathcal{I} \leq_{RK} \mathcal{J}$ and $\mathcal{J}$ satisfies FinBW. Then $\mathcal{I}$ satisfies FinBW.*

Proof. Let $f\colon \omega \to \omega$ be such that $A \in \mathcal{I}$ iff $f^{-1}(A) \in \mathcal{J}$, and $(x_n) \subset [0,1]$. Let $x'_n = x_{f(n)}$ for each n. By FinBW property of $\mathcal{J}$ there exists $A' \notin \mathcal{J}$ such that $(x'_n) \restriction A'$ is Fin-convergent to an $x \in [0,1]$. Let $A = f(A')$. Since $A' \subset f^{-1}(A)$ and $A' \notin \mathcal{J}$, $A \notin \mathcal{I}$. Since $(x'_n)_{n\in A'}$ is Fin-convergent, for every $\varepsilon > 0$ the set

$$T = \{n \in A' : |x'_n - x| \geq \varepsilon\} = \left\{n \in A' : \left|x_{f(n)} - x\right| \geq \varepsilon\right\}$$

is finite, hence

$$f(T) = \left\{f(n) \in A : \left|x_{f(n)} - x\right| \geq \varepsilon\right\} = \{m \in A : |x_m - x| \geq \varepsilon\}$$

is finite. Thus $(x_n) \restriction A$ is Fin-convergent. □

The next argument shows that we cannot replace FinBW with BW in Theorem 6.1.

Let $\mathcal{I} = \mathcal{I}_d$ and $\mathcal{J} = \mathcal{I}_d \oplus \mathcal{I}_m$, where $\mathcal{I}_m$ is a maximal extension of $\mathcal{I}_d$. Clearly $\mathcal{I} \notin \mathrm{BW}$ and $\mathcal{J} \in \mathrm{BW}$. We may assume that $\mathcal{J}$ is an ideal defined on the set $\omega \times \{0,1\}$, with $\mathcal{J} \restriction \omega \times \{0\}$ isomorphic to $\mathcal{I}_d$ and $\mathcal{J} \restriction \omega \times \{1\}$ isomorphic to $\mathcal{I}_m$. Let $f\colon \omega \times \{0,1\} \to \omega$ be a projection of $\omega \times \{0,1\}$ on ω, i.e. $f(n,b) = n$ for every

$n \in \omega$, $b \in \{0,1\}$. It is easy to see that f is 2-to-one and $A \in \mathcal{I}$ iff $f^{-1}(A) \in \mathcal{J}$. Thus $\mathcal{I} \leq_{RK} \mathcal{J}$.

Recall that an ideal $\mathcal{I}$ on ω is a *Q-ideal* if for every partition $(A_n)_{n\in\omega}$ of ω into finite sets there is $S \in \mathcal{I}^*$ with $|S \cap A_n| \leq 1$ for every $n \in \omega$ (see e.g. [3] where Q-ideals are called Q-points.)

Theorem 6.2. *Let $\mathcal{I} \leq_{RB} \mathcal{J}$ and $\mathcal{J}$ be a Q-ideal. If $\mathcal{J}$ satisfies hBW then $\mathcal{I}$ satisfies hBW.*

Proof. Let $f : \omega \to \omega$ be finite-to-one with $A \in \mathcal{I} \iff f^{-1}(A) \in \mathcal{J}$. Since $\mathcal{J}$ is Q-ideal there is $S \in \mathcal{J}^*$ with $f \restriction S$ is one-to-one. Let (x_n) be a bounded sequence and $A \notin \mathcal{I}$. Since

$$A' = f^{-1}(A) \notin \mathcal{J},$$

so

$$B' = A' \cap S \notin \mathcal{J}.$$

The ideal $\mathcal{J}$ satisfies hBW hence there is $C' \subset B'$ such that $C' \notin \mathcal{J}$ and $(x_{f(n)}) \restriction C'$ is $\mathcal{J}$-convergent (say, $\mathcal{J} - \lim(x_{f(n)}) \restriction C' = x$).

Let $C = f(C') \notin \mathcal{I}$. Let

$$X = \left\{n \in C' \ : \ \left|x_{f(n)} - x\right| > \varepsilon\right\}.$$

Then $X \in \mathcal{J}$. We will show that $(x_n) \restriction C$ is $\mathcal{I}$-convergent. It is enough to show that $f(X) \in \mathcal{I}$. For the sake of contradiction, suppose that $f(X) \notin \mathcal{I}$. Then $Y = f^{-1}(f(X)) \notin \mathcal{J}$ hence $Y \cap S \notin \mathcal{J}$ but on the other hand $Y \cap S = X$, a contradiction. □

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