

The most important results from my thesis:

- It is consistent with ZFC that there exists (s) -measurable additive function which is discontinuous.
- There exist two (s_0) -sets whose algebraic sum is not (s) -measurable.
- It is consistent with ZFC that the family of functions with the Baire property has the difference property.
- The family of (s) -measurable functions does not have the difference property.
- If we assume Continuum Hypothesis, then there exists a function which has Borel measurable difference functions but of unlimited Baire class.
- Families of measurable functions do not have (“usually”) the difference property (at least under some set-theoretic assumptions).

A function $A: \mathbb{R} \rightarrow \mathbb{R}$ is called *additive* if it satisfies Cauchy’s functional equation: $A(x + y) = A(x) + A(y)$ for every $x, y \in \mathbb{R}$. It is known that every continuous additive function is of the form $A(x) = ax$ for some $a \in \mathbb{R}$. On the other hand, Hamel showed that there are discontinuous additive functions.

For any function $f: \mathbb{R} \rightarrow \mathbb{R}$ and $h \in \mathbb{R}$ we define the *difference function* $\Delta_h f: \mathbb{R} \rightarrow \mathbb{R}$ by formula $\Delta_h f(x) = f(x + h) - f(x)$. It is easy to show, that all difference functions of an additive function are continuous.

Taking the additive function constructed by Hamel, we can say that there exist discontinuous functions with continuous difference functions. Moreover, Ostrowski proved, that the function constructed by Hamel is not Lebesgue measurable, therefore there even are Lebesgue nonmeasurable functions with continuous difference functions.

The following question arise naturally: what can one say about functions whose all difference functions are continuous? Erdős conjectured that such a function is a sum of two functions: a continuous function and an additive one. In 1951, Erdős’s conjecture was proved by de Bruijn in the paper “*Functions whose differences belong to a given class*”.

Theorem 1 (de Bruijn). *If $\Delta_h f$ is continuous for every $h \in \mathbb{R}$, then $f = g + A$, where $g: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $A: \mathbb{R} \rightarrow \mathbb{R}$ is an additive function.*

In the same article, de Bruijn introduced the notion of the *difference property*.

Definition 1. A family of real functions \mathcal{F} has the *difference property*, if every function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $\Delta_h f \in \mathcal{F}$ for each $h \in \mathbb{R}$ is of the form $f = g + A$, where $g \in \mathcal{F}$ and $A: \mathbb{R} \rightarrow \mathbb{R}$ is an additive function.

Moreover, de Bruijn proved the difference property for families of differentiable functions, continuously differentiable functions, and others. Since de Bruijn introduced the difference property, many mathematicians have been interested in this subject. A good source of information on the difference property is Laczkovich's survey "*The difference property*".

In Chapter 2, we show some results about properties of additive functions and algebraic sums of measurable sets. For instance, we show that there exists (s) -measurable additive function which is discontinuous. It is known that there is no Lebesgue measurable counterpart of the above result. Moreover, we construct a set $A \in (s_0)$ such that $A + A$ is not (s) -measurable. In that case we get similar result as for Lebesgue measure. Results from this chapter are contained in the following papers: "*A note on algebraic sums of Marczewski measurable sets*" and "*Total Failure of Automatic Continuity for Marczewski Measurable Functions*", written with F. Darais.

Chapter 3 concerns the difference property for the family of functions with the Baire property. We solved the following problem posed by Laczkovich.

Problem 1 (Laczkovich). Is it consistent with ZFC, that the family of functions with the Baire property has the difference property?

Results from this chapter were published in the article "*On the difference property of the family of functions with the Baire property*".

In Chapter 4 we work with Marczewski measurable functions. We prove that the family of Marczewski measurable functions does not have the difference property. This theorem is contained in the paper "*On the Difference Property of Borel Measurable and (s) -Measurable Functions*", written with I. Reclaw.

In Chapter 5 we consider Borel measurable functions. Working on the difference property for that family, Laczkovich posed the following problem:

Problem 2 (Laczkovich). Suppose that all difference functions of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ are Borel measurable. Does there exist $\alpha < \omega_1$ such that all difference functions of f are Baire class α ?

We proved that under Continuum Hypothesis the answer is no. The proof was published in "*On the Difference Property of Borel Measurable and (s) -Measurable Functions*", written with I. Reclaw.

In Chapter 6 we prove some general theorems concerning the difference property for families of measurable functions. Then we use them to examine the difference property for some families of measurable functions. Results from this chapter were published in "*On the difference property of families of measurable functions*".